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Effective medium theory of the one-dimensional resonance phononic crystal

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Abstract
A general theoretical scheme to describe the effective modulus and mass density for acoustic metamaterials is presented. For such a purpose, an effective medium theory of a one-dimensional acoustic waveguide containing subwavelength-sized Helmholtz resonators is formulated. It is shown that, when the wavelength is much larger than the periodic length and the size of the resonators, the whole composite structure can be treated as an effective homogeneous medium in accounting for its acoustic properties. It is also shown that the acoustic characteristics, such as the effective modulus and the effective mass density, can be determined precisely from the transmission and the reflection data. The calculated effective modulus and effective mass density confirm that this structure behaves as a homogeneous metamaterial with a negative effective modulus in a frequency range just above the resonant frequency.

The study of acoustic and elastic wave propagation in phononic crystals [1, 2] has received increasing attention in the last few decades [3–14]. More recently, unusual acoustic wave phenomena related to acoustic superlenses, such as negative refraction or wave focusing effects, have been attracting much attention [9–11]. A superlens can produce images that contain details finer than the wavelength of the original waves. The negative refractive behaviour for the acoustic wave can be described by introducing a negative effective density and/or a negative refractive index [11]. Although the static elastic constant must be positive to maintain structural stability, an effective dynamic medium may possess resonance-induced negative dynamic parameters [6, 8, 12–14]. The design of a material with an effective negative mass density and/or an effective negative modulus, which is called an acoustic metamaterial, has been demonstrated theoretically and experimentally [6, 8, 9, 11–14].

In view of the rapid progress in acoustic metamaterials, it is imperative to clarify the concept and provide a general theoretical scheme to describe correctly the effective modulus and mass density of the metamaterials. At present, people infer the existence of the negative modulus from analysis of the dispersion relation in given structures. Some works [6, 8] have suggested that a negative group velocity is sufficient to obtain a negative effective modulus. For a dispersive acoustic material with a phase speed, c, and a wavenumber, $k = \omega/c$, we express its group velocity as

$$v_g = \frac{d\omega}{dk(\omega)} = \frac{c}{1 - \frac{\omega}{c}\frac{dc}{d\omega}}.$$

Since the acoustic phase speed is $c = \sqrt{\frac{E}{\rho}}$, where E and ρ are the modulus and the mass density of the material being considered respectively, it is clear that the modulus or the density cannot uniquely be determined by v_g. Therefore, a more rigorous scheme is required to characterize acoustic metamaterials precisely.

The aim of this work is to establish a general theoretical scheme to determine the effective modulus and effective mass density of acoustic metamaterials uniquely

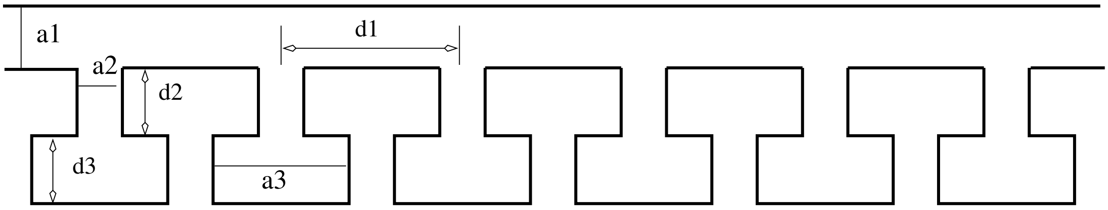

Figure 1. Schematic cross-sectional view of the geometry of the structure considered in this work. a_1, a_2 and a_3 represent the cross-section of the slender waveguide, the neck and the cavity tubes, respectively. d_1 is the periodic length of the structure along the slender tube. The hole has a length d_2, and the length of the cavity is d_3.

from the transmission and reflection coefficients for one-dimensional (1D) acoustic crystals. In electromagnetism, to characterize the physical properties of metamaterials correctly, researchers [15–17] have developed methods for retrieving the effective permittivity and permeability from known transmission and reflection coefficients. The main idea of these methods is simple: first the finite-size 1D photonic crystal to be treated is regarded as a homogeneous slab with the same width and, then, using the transmission and reflection coefficients of the electromagnetic waves through the finite crystal as those of the slab, one calculates the effective permittivity and permeability of the slab. The detailed process is as follows. For a homogeneous slab with a width d, it is known that the transmission and the reflection coefficients are

$$t = \left[\cos(nkd) - \frac{\mathrm{i}}{2}\left(z + \frac{1}{z}\right)\sin(nkd)\right]^{-1}, \tag{1}$$

$$r = \frac{-\frac{\mathrm{i}}{2}\left(z - \frac{1}{z}\right)\sin(nkd)}{\cos(nkd) - \frac{\mathrm{i}}{2}\left(z + \frac{1}{z}\right)\sin(nkd)}, \tag{2}$$

where n is the relative index, z is the relative impedance and k is the wavevector of the normally incident wave in vacuum. Inversely, n and z can be determined as,

$$z = \pm\sqrt{\frac{(1+r)^2 - t^2}{(1-r)^2 - t^2}}, \tag{3}$$

$$n = \pm\frac{\arccos\frac{1-(r^2-t^2)}{2t}}{kd} + \frac{2m\pi}{kd}. \tag{4}$$

Once n and z are obtained, the permittivity ε and the permeability μ can be calculated directly from $\mu = nz$ and $\varepsilon = n/z$. We should choose the sign which meets the requirement $\mathrm{Re}(z) \geqslant 0$ and $\mathrm{Im}(n) \geqslant 0$. The branch $\mathrm{Re}(n)$ or the integer m is determined by the condition that ε and μ are continuous functions of frequency.

Although the above methods are originally conceived for electromagnetic waves, it should be valid for any wave phenomenon because the key characteristic parameters, namely the relative wave impedance and the relative index, are universal for the wave phenomena. For acoustic wave propagation through a homogeneous slab inside a reference material (like vacuum for electromagnetic waves), the relative refractive index, n, is defined as $n = \frac{c_0}{c_{\mathrm{slab}}}$, and the relative impedance as $z = \frac{Z_{\mathrm{slab}}}{Z_0}$. c_0 and Z_0 are the acoustic phase speed and the impedance of the reference, respectively. Thus, when the transmission t and the reflection r are known, one can obtain c_{slab} and Z_{slab}, and hence the acoustic modulus, $E = Zc$, and the mass density, $\rho = Z/c$.

In the following, we demonstrate the scheme for determining the effective modulus and the effective mass density by analytically investigating a specific example, as shown in figure 1. The model is a 1D waveguide composed of a continuous slender tube with the periodically jointed Helmholtz resonators [8, 18, 19]. The Helmholtz resonator consists of two tube segments with different lengths and cross sections. The cavity tube has a length d_3 and a cross-section a_3 (its volume is $V_3 = a_3 d_3$) and the connecting neck has a length d_2 and a cross-section a_2 (its volume is $V_2 = a_2 d_2$). The cross-section of the main slender tube is a_1 and the periodic length is d_1 (its volume is $V_1 = a_1 d_1$).

It is known in the framework of interface response theory [20] that the inverse surface Green's function for a semi-infinite waveguide tube is

$$[g_{\mathrm{s}}]^{-1} = -\frac{\mathrm{i}}{Z}, \tag{5}$$

where $Z = \frac{\rho c}{a}$ is the impedance of the tube [21], and the inverse surface Green's function of a finite tube with a length d under the closed boundary condition is

$$g_{\mathrm{f}}^{-1} = \frac{1}{Z\sin(\alpha d)}\begin{pmatrix} -\cos(\alpha d) & 1 \\ 1 & -\cos(\alpha d)\end{pmatrix}, \tag{6}$$

where $\alpha = \omega/c$, and ω is the angular frequency of the incident acoustic wave. As each Helmholtz resonator is a combination of two finite tubes under the closed boundary condition, its inverse Green's function can be represented as a simple addition of those of tube 2 and tube 3. Since the Helmholtz resonator contacts the waveguide tube at the free end of the neck, here we only give the element of the inverse surface Green's function matrix on the free end of tube 2:

$$g_{\mathrm{r}}^{-1} = \frac{\frac{Z_3}{Z_2} + \cot(\alpha_2 d_2)\tan(\alpha_3 d_3)}{Z_3\cot(\alpha_2 d_2) - Z_2\tan(\alpha_3 d_3)}. \tag{7}$$

The inverse surface Green's function of the composite system is then obtained as an infinite-dimensional matrix defined in the interface domain consisting of all the connection points. The diagonal and off-diagonal elements of this matrix are, respectively, given by $[-\frac{2}{Z_1}\cot(\alpha_1 d_1) + M g_{\mathrm{r}}^{-1}]$ and $[\frac{1}{Z_1}\sin(\alpha_1 d_1)]$. Taking advantage of the translational

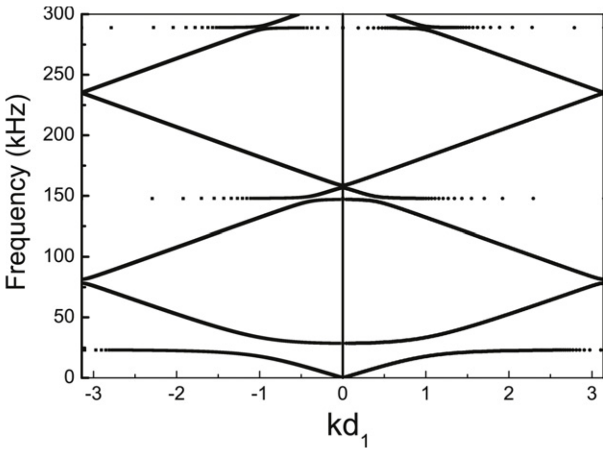

Figure 2. The acoustic band structure for the model with water inside the waveguide and the Helmholtz resonators. The geometric parameters [8] are $a_1 = 16$ mm^2, $d_1 = 9.2$ mm, $a_2 = 0.25\pi$ mm^2, $d_2 = 1$ mm, $a_3 = 4\pi$ mm^2, $d_3 = 5$ mm. The mass density of water is $\rho_1 = \rho_2 = \rho_3 = 1.0 \times 10^3$ kg m^{-3}, and the sound speed in water is $c_1 = c_2 = c_3 = 1.485 \times 10^3$ m s^{-1}.

periodicity along the infinitely extended waveguide tube, the dispersion relation of the model can be obtained [22],

$$\cos(kd_1) = \cos(\alpha_1 d_1) - \frac{M}{2} Z_1 \sin(\alpha_1 d_1) g_r^{-1}, \qquad (8)$$

where k is the complex Bloch propagation vector along the infinitely extended waveguide tube. The acoustic band structure is shown in figure 2 in the case of an identical material inside the slender tube and the resonators. There exist two types of gap: one is the conventional Bragg band gap due to the periodicity of the slender tube, and the other is the resonant band gap originating from the resonances of the individual branches. The central frequency of the Bragg gap satisfies $\sin(\frac{\omega d_1}{c_1}) = 0$, while the central frequency of the resonant gap satisfies

$$Z_3 - Z_2 \tan\left(\frac{\omega d_2}{c_2}\right) \tan\left(\frac{\omega d_3}{c_3}\right) = 0. \qquad (9)$$

For a suitable choice of the geometrical sizes of the periodic unit, the width of the lowest gap that is a resonant gap becomes the largest one among all the gaps, and the geometrical sizes become much smaller than the corresponding wavelength in the lowest gap. In the case of the same filling material in all parts of the structure, using the approximation $\sin(\alpha_i d_i) \simeq \alpha_i d_i$ and $\cos(\alpha_i d_i) \simeq 1 - \frac{1}{2}(\alpha_i d_i)^2$, we get the central frequency of the lowest gap from equation (9),

$$\omega_0 = c\sqrt{\frac{a_2}{V_3 d_2}}, \qquad (10)$$

and the frequency range of the lowest gap from equation (8),

$$\left[1 + \frac{V_2 + V_3}{4V_3}\frac{a_2 d_1}{a_1 d_2}\right]^{-1/2} \leqslant \frac{\omega}{\omega_0} \leqslant \left[1 + \frac{V_2 + V_3}{V_1}\right]^{1/2}, \qquad (11)$$

where V_i is the volume of the tube i.

Since the wavelength is much larger than the periodic length and the size of the resonators in the lowest gap, it is valid to replace the whole composite by an effective homogeneous medium in considering its acoustic properties [13]. For a finite structure with N cells connected at its ends to two semi-infinite leading tubes, in the interface response theory the inverse surface Green's function, G^{-1}, of the composite system is given by a $[(N+1) \times (N+1)]$ matrix defined in the interface domain consisting of all the connection points. The diagonal and off-diagonal elements of this matrix are, respectively, given by $[-\frac{2}{Z_1}\cot(\alpha_1 d_1) + g_r^{-1}]$ and $[\frac{1}{Z_1}\sin(\alpha_1 d_1)]$, except for the top-left and the bottom-right elements, which change to $[-\frac{i}{Z_1} - \frac{1}{Z_1}\cot(\alpha_1 d_1) + g_r^{-1}]$ and $[-\frac{i}{Z_1} - \frac{1}{Z_1}\cot(\alpha_1 d_1)]$, respectively. Then the transmission coefficient t and the reflection coefficient r can be obtained directly,

$$t = \frac{2i}{Z_1} G(0, N), \qquad r = -1 + \frac{2i}{Z_1} G(0, 0), \qquad (12)$$

where $G(0, N)$ and $G(0, 0)$ are the top-right and the top-left elements of the surface Green's function matrix, respectively.

Substituting equation (12) into equation (3), we get the effective z and n representing the equivalent impedance and refractive index of a homogeneous slab which has the same width as the finite structure. Finally, we obtain the effective modulus and mass density of this composite structure. Because of the localized nature of the resonances, sonic attenuation is apparent near the resonant frequency even for a structure with a single unit which lacks periodicity. Also, in electromagnetism, the calculated reflection and transmission for multiple cells using the retrieved n and z from the single-cell data are known to match well with the reflection and transmission data for the multiple-cell case [15, 17]. Thus, here we calculate the effective relative modulus and density only with the single unit cell structure.

The results are shown in figure 3. Since we are only interested in the properties of the model in the frequency range around the lowest gap, figure 3(a) shows the transmission amplitude of a ten-unit finite structure in the low-frequency domain. In the lowest gap, the maximal attenuation that corresponds to the resonant frequency of the Helmholtz resonator is located near the lower edge of the gap. The frequency dependence of the dispersion, the effective bulk modulus and the effective bulk mass density are shown in figures 3(b)–(d), respectively. When the frequency is small enough, the acoustic wave transmits through the structure completely and, in this frequency range, the wavevector, the effective modulus and the effective density are all real. The ratio, k/ω, and the effective parameters are mostly positive. This means that the finite structure can be treated as a non-dispersive homogeneous material when the wavelength is much larger than the structure length. When the incidental frequency tends to the gap gradually, mini-gaps appear in the transmission spectra, and the imaginary parts of the wavevector, of the effective modulus and of the effective density become nonzero and increase, whereas $\mathrm{Re}(E_{\mathrm{eff}})$,

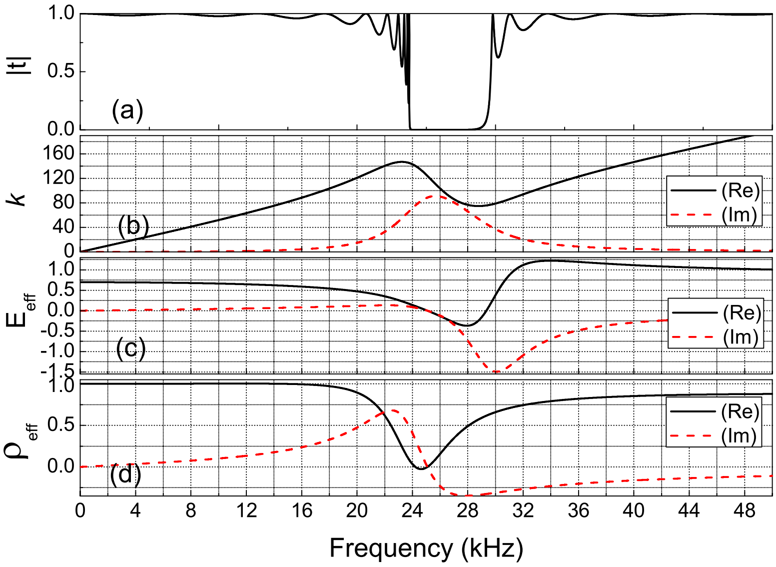

Figure 3. Calculated effective modulus and mass density near the resonant frequency for the structure shown in figure 1. The model parameters are the same as those in figure 2: (a) the transmission spectra, (b) the dispersion curve, (c) the effective modulus, and (d) the effective mass density.

(This figure is in colour only in the electronic version)

$\mathrm{Re}(\rho_{\mathrm{eff}})$ and the slope of $\mathrm{Re}(k)$ begin to decrease. Inside the gap, the slope of $\mathrm{Re}(k)$ is negative, thus implying a negative group velocity. As the frequency of the incident wave approaches the resonant frequency of the Helmholtz resonator, the effective dynamic parameters change drastically. The effective mass density reaches its minimum at a frequency which is slightly lower than the resonant frequency of the Helmholtz resonator, and then increases gradually. The effective modulus decreases gradually and crosses the zero point at the resonant frequency. When the frequency is higher than the resonant frequency inside the gap, $\mathrm{Re}(E_{\mathrm{eff}})$ is negative, while $\mathrm{Re}(\rho_{\mathrm{eff}})$ is mostly positive. When the frequency exceeds the gap, the wavevector, the effective modulus, and the effective density become real and positive again, and the slope of k tends to a new positive constant.

In addition, from the effective medium approximation [23], we also obtain, in the steady limit,

$$E_{\mathrm{eff}}(\omega \approx 0) = E_0 \frac{1}{1 + \frac{V_2+V_3}{V_1}}, \tag{13}$$

and

$$\rho_{\mathrm{eff}}(\omega \approx 0) = \rho_0, \tag{14}$$

where E_0 and ρ_0 are the modulus and the density of the reference material. Comparing with the formula [8]

$$E_{\mathrm{eff}}^{-1} = E_0 \left[1 - \frac{F\omega_0^2}{\omega^2 - \omega_0^2 + \mathrm{i}\Gamma\omega}\right], \tag{15}$$

we find the geometrical factor

$$F = \frac{V_2 + V_3}{V_1}. \tag{16}$$

But the damping parameter Γ cannot be determined *a priori*, and, thus, should be determined through the experiments.

It is interesting to observe that the lowest resonant gap is divided into two different regions with different characters at the resonant frequency. Although the group velocity is negative inside the whole gap, in a narrow frequency range below the resonant frequency, the effective mass density has a negative real part; there is a broad range for an effective negative modulus up to the resonant frequency. For a given system the same as that of [8], we find that the real part of the effective mass density is negative in the frequency range (24.21, 25.08) kHz and that the real part of the effective modulus is negative in the frequency range (25.08, 29.35) kHz. The value of 25.08 kHz is just the resonant frequency obtained by equation (7). Although both the modulus and the mass density have imaginary parts inside the resonant gap, this fact does not mean any absorption or gain, since all the constituent blocks of the model are conventional acoustic materials [24].

In summary, we have presented a theoretical scheme to characterize an acoustic metamaterial from the transmission and reflection coefficients. This method allows us to determine the dispersion relation, the effective modulus and the effective mass density separately. The method is applied to an acoustic waveguide containing periodic Helmholtz resonators. It is shown that this structure behaves as a negative effective modulus bulk material in the low-frequency range. Moreover, since the resonant gap is a function of the sizes and pattern of the structural unit, one can tune the effective elastic modulus to negative values at desired frequency ranges.

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